Asymptotic linearization of the Fisher equation for a class of initial conditions

Sanjay Puri†‡ and Alan J Bray§

† School of Physical Sciences, Jawaharlal Nehru University, New Delhi–110067, India
‡ Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
§ Department of Theoretical Physics, The University, Manchester M13 9PL, UK

Abstract. We present a novel method of linearizing the Fisher equation asymptotically in time, i.e. we use a nonlinear transformation to convert the initial value problem for the Fisher equation into an initial value problem for the diffusion equation for a class of initial conditions. The analytic result we obtain is in excellent agreement with numerical simulations. Our method easily generalizes to other reaction–diffusion equations also.

Many important physical phenomena are described by nonlinear partial differential equations. In particular, two classes of nonlinear partial differential equations have been the subject of considerable theoretical and numerical interest, namely:

(i) those arising from completely integrable, infinite-dimensional Hamiltonian systems [1]; and

(ii) the so-called reaction–diffusion equations, which combine the effects of a local reaction with those of spatial diffusion [2].

For the first class of nonlinear partial differential equations mentioned above, there has been much progress towards solving the initial value problem for arbitrary initial conditions. Powerful techniques like the inverse scattering transform and Backlund transformations enable the solution of these nonlinear equations by reducing the problem to one involving a sequence of linear equations [1]. The reduction of the problem of solving a nonlinear partial differential equation to the problem of solving related linear equations is often referred to as a linearization of the nonlinear partial differential equation. A particularly well known example is that of the Hopf–Cole transform [3], which linearizes Burgers' equation [4] (also in the first class mentioned above) by a nonlinear transformation of the dependent variable. The resultant linear equation is merely the diffusion equation.

There has been considerably less progress in solving the general initial value problem for reaction–diffusion equations. For a scalar field, these have the general form

$$u_t(r, t) = f(u(r, t)) + \nabla^2 u(r, t) \tag{1}$$

where $u(r, t)$ is an order parameter field (e.g. population density, magnetization, chemical concentration) defined as a function of space (r) and time (t). The function $f(u(r, t))$ is usually a nonlinear function which models the local reaction. Particular interest has focused upon the cases $f(u) = u - u^2$ (namely, the Fisher equation, which describes the dynamics of geographically structured populations [5]) and $f(u) = u - u^3$ (namely, the

time-dependent Ginzburg–Landau or TDGL equation, which describes the dynamics of the order–disorder transition [6]). For the one-dimensional (1D) case, Aronson and Weinberger [7] have demonstrated that a broad class of initial conditions for (1) (under certain weak constraints on $f(u)$) converge to a travelling-wave kink solution with a unique velocity. In general, the analytic form of this solution (referred to as a 'cline') is as yet unknown, to the best of our knowledge. In arbitrary dimensions, Kawasaki *et al* [8] have used singular perturbation techniques to obtain an approximate asymptotic solution to the initial value problem for the TDGL equation. Because of the large number of approximations involved, their solution is not a good approximation to the actual solution (obtained numerically)†.

A similar approach has also been applied by Puri *et al.* [11] to the Fisher equation in arbitrary dimensions. Again, the resultant analytic solution is not a good approximation to the actual solution (obtained numerically) though it does approach the correct asymptotic velocity.

The singular perturbation approach of Kawasaki *et al.* [8] and Puri *et al.* [11] is equivalent to nonlinear transformations of the dependent variable $u(r, t)$ which (after a somewhat unjustifiable approximation) transform the TDGL or Fisher equations into the diffusion equation. As already pointed out, the untenable approximations involved result in an analytic solution that does not closely mimic the actual solution. In this paper, we present a novel method of linearizing reaction–diffusion equations. Our linearization is asymptotically exact. Unfortunately, as we will see shortly, the linearization is only valid for a certain class of initial conditions. However, for representative initial conditions in this classs, the agreement of our analytic solution with the numerical results is excellent. We present our linearization scheme in the context of the Fisher equation here. However, our approach readily generalizes to other reaction–diffusion equations also.

As we mentioned earlier, the Fisher equation in arbitrary dimensions has the form

$$u_t(r, t) = u(r, t) - u(r, t)^2 + \nabla^2 u(r, t). \tag{2}$$

For the Fisher equation, only solutions with $u(r, t) \geq 0$ are of physical interest as (2) is unstable for $u(r, t) < 0$. Our transformation of the dependent variable $u(r, t)$ is motivated by the singular perturbation result for the initial value problem of the Fisher equation [11], namely

$$u(r, t) = \frac{u^0(r, t)}{1 + u^0(r, t)} \tag{3}$$

where $u^0(r, t)$ satisfies the linear part of the Fisher equation, namely $u^0(r, t) = e^{t(1+\nabla^2)}u(r, 0)$. We consider the nonlinear transformation

$$u(r, t) = \frac{e^t f(r, t)}{1 + e^t f(r, t)}. \tag{4}$$

This transformation would seem to imply that we cannot have $u(r, t) = 1$ for any finite times as that would correspond to $f(r, t) = \infty$. However, as we see shortly it turns out that $1/f(r, t)$ (and not $f(r, t)$) satisfies the diffusion equation asymptotically in time.

---

† We should point out, however, that their solution provides a good statistical description of the defects (interfaces) in the system and gives excellent results for the time-dependent structure factor. This fact has been exploited by Puri [9] and Bray and Puri [10] to calculate the time-dependent structure factor for the 2-component and n-component TDGL equations, respectively.

Thus, we need to impose the constraint that $u(r, t) \neq 0$ rather than $u(r, t) \neq 1$ for all finite times. From the nonlinear transformation (4), we find that the partial differential equation satisfied by $f(r, t)$ is

$$f_t(r, t) = \nabla^2 f(r, t) - \frac{2[\nabla f(r, t)]^2}{e^{-t} + f(r, t)}. \tag{5}$$

The singular perturbation approach is equivalent to neglecting the nonlinear term in (5) so that $f(r, t)$ satisfies the diffusion equation. But this is an unjustified approximation and one can actually do much better, as we demonstrate. If we restrict ourselves to functions $f(r, t)$ which are non-zero as $t \to \infty$, we can neglect e^{-t} in the denominator of the nonlinear terms of (5) for long times. Therefore, asymptotically in time, (5) becomes

$$f_t(r, t) = \nabla^2 f(r, t) - \frac{2[\nabla f(r, t)]^2}{f(r, t)} \tag{6}$$

which is equivalent (after dividing both sides by $f(r, t)^2$) to

$$\left(\frac{1}{f(r, t)} \right)_t = \nabla^2 \left(\frac{1}{f(r, t)} \right). \tag{7}$$

Thus, the variable $1/f(r, t)$ obeys the diffusion equation and we have been successful in linearizing the Fisher equation for a class of initial conditions (asymptotically in time). The restriction on the initial conditions arises because the above procedure becomes invalid if $f(r, t)$ becomes 0 at any point. We have

$$[f(r, t)]^{-1} = e^{t\nabla^2} \left[\frac{1 - u(r, 0)}{u(r, 0)} \right] \tag{8}$$

and, therefore, $u(r, 0)$ cannot take the value 0 anywhere. Thus, our linearization is only valid for the class of initial conditions which lie uniformly above 0. For this class of initial conditions, the asymptotic solution to the initial value problem for the Fisher equation is

$$u(r, t) = \frac{e^t}{e^{t\nabla^2} \left[\dfrac{1 - u(r, 0)}{u(r, 0)} \right] + e^t}. \tag{9}$$

Clearly, the most interesting class of initial conditions for Fisher equation consist of profiles which go through 0 as these are the initial conditions which evolve into travelling-wave (cline) solutions. The class of initial conditions considered here translate uniformly in time towards the fixed point $u^* = 1$. Nevertheless, our linearization is of considerable methodological interest. Furthermore, as we will see shortly, our analytic solution is in excellent agreement with numerical results.

Before presenting our numerical results, we remark that (9) gives the exact result of homogenous initial conditions, namely

$$u(t) = \frac{e^t u(0)}{1 - u(0) + e^t u(0)}. \tag{10}$$

This is in contrast to the singular perturbation results in (3), which gives an incorrect result for the homogenous case. For arbitrary initial conditions which lie
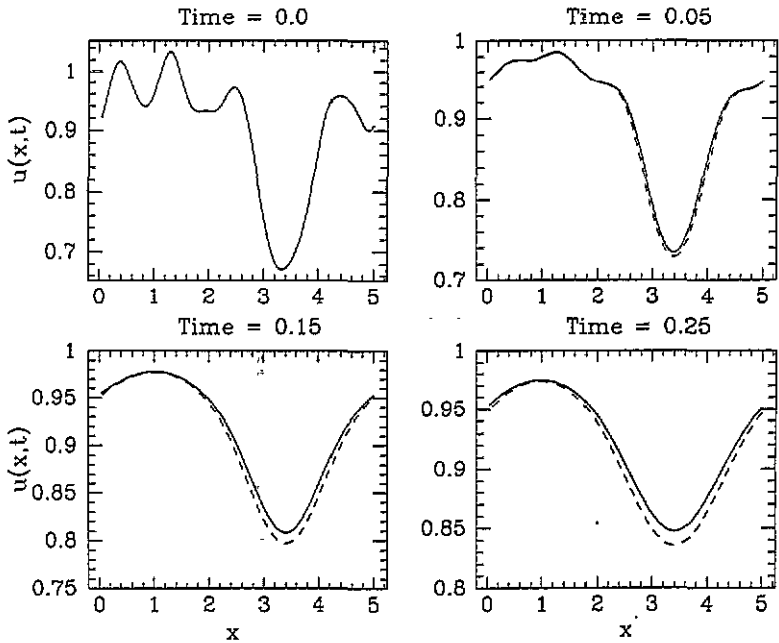
Figure 1. Comparison of results from a numerical simulation of the one-dimensional Fisher equation with our analytic expressions. The simulation was done using a simple Euler discretization scheme on a lattice of length $L=5$ with periodic boundary conditions. The mesh sizes were $\Delta x = 0.05$ and $\Delta t = 0.00025$. The initial condition (shown in the first frame) consists of uniformly distributed random fluctuations of amplitude 0.2 about a background value of 0.85. Results are shown in three different frames for times 0.05, 0.15 and 0.25 with the solid line representing the numerical solution; the dashed line representing the analytic solution from (9) in the text; and the dotted line (which is barely distinguishable from the numerical result) representing a first-order perturbation result with the analytic solution of (9) as its zeroth-order term.

uniformly above 0, we have compared the results of a 1D numerical simulation with our analytic result (9) in figure 1. The initial condition (labelled as 'Time = 0.0') consists of uniform random fluctuations of amplitude 0.2 about a background value of 0.85. Our simulation was performed using a simple Euler discretization scheme on a lattice of length $L=5$ with periodic boundary conditions. The mesh sizes of our simulation was $\Delta x = 0.05$ and $\Delta t = 0.00025$. Figure 1 shows the results of our simulation as a solid line for times 0.05, 0.15 and 0.25. The analytic form of (9) is denoted by the dashed curve in the different frames and the agreement with the numerical result is seen to be very good. We can make the agreement almost perfect by treating e^{-t} as a small term in (5) and carrying out a first-order perturbation expansion for $1/f(r, t)$ as follows. If we define $g(r, t) = 1/f(r, t)$, (5) becomes

$$g_t(r, t) = \nabla^2 g(r, t) - 2\,e^{-t}\frac{[\nabla g(r, t)]^2}{1+e^{-t}g(r, t)}. \tag{11}$$

If we expand the denominator of the second term on the right-hand side of (11), we obtain

$$g_t(r, t) = \nabla^2 g(r, t) - 2\,e^{-t}[\nabla g(r, t)]^2 \sum_{h=0}^{\infty} (-1)^n\,e^{-nt}g(r, t)^n. \tag{12}$$

A systematic perturbative improvement of our result (8) for $1/f(r, t)$ is obtained in the usual fashion by introducing a parameter λ (which we will subsequently set equal to 1) and expanding $g(r, t)$ as

$$g(r, t) = \sum_{h=0}^{\infty} \lambda^n g^{(n)}(r, t). \tag{13}$$

Replacing this in (12) and matching terms in each order, we find that $g^{(0)}(r, t)$ satisfies the diffusion equation (as expected) and $g^{(1)}(r, t)$ satisfies the diffusion equation with a source term, namely

$$g_t^{(1)}(r, t) = \nabla^2 g^{(1)}(r, t) - 2 e^{-t} [\nabla g^{(0)}(r, t)]^2$$

$$\equiv \nabla^2 g^{(1)}(r, t) - F(r, t). \tag{14}$$

This equation is easily solved using Green's functions as

$$g^{(1)}(r, t) = - \int dr' \int_0^t dt' \frac{1}{[4\pi(t-t')]^{d/2}} \exp\left(-\frac{(r-r')^2}{4(t-t')}\right) F(r', t') \tag{15}$$

where d is the dimensionality and we set $g^{(1)}(r, 0) = 0$. The corresponding first-order perturbation theory result for $u(r, t)$ is obtained from the nonlinear transformation (4) with $f(r, t) = 1/[g^{(0)}(r, t) + g^{(1)}(r, t)]$.

In figure 1, we plot the analytic result from this first-order perturbation theory as a dotted line. It is almost indistinguishable from the numerical solution. Figure 2 shows the absolute value of the differences (denoted by $e(x, t)$) between
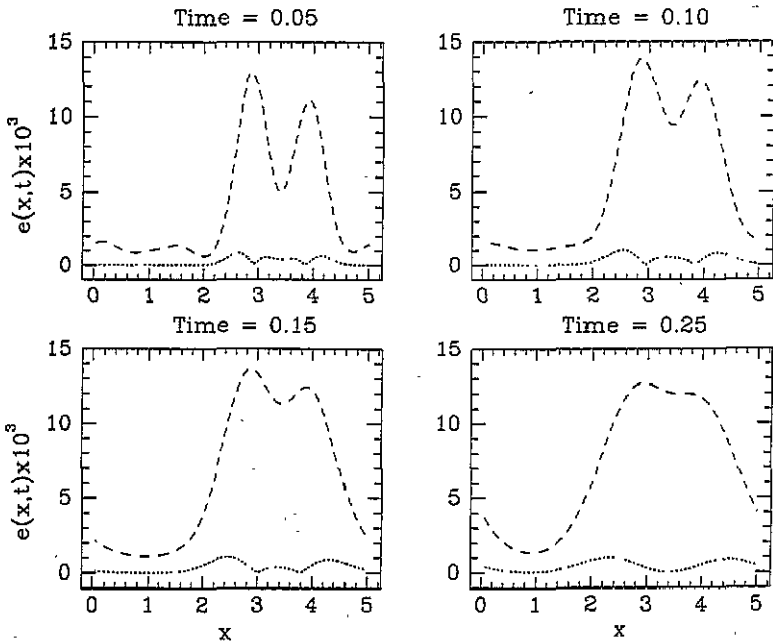

Figure 2. Modulus of differences $e(x, t)$ versus x between the numerical solution and our analytic solution from (9) (denoted by a dashed line); and between the numerical solution and the result from first-order perturbation theory around the solution from (9) (denoted by a dotted line). The simulation is the same as that for figure 1 and we show results for times 0.05, 0.10, 0.15 and 0.25.

(i) the actual solution and the analytic form from (9) (denoted by the dashed line);
and

(ii) the actual solution and the first-order perturbation theory result (denoted by
the dotted line).

The maximum error associated with the analytic solution of (9) is $O(10^{-2})$. On the
other hand, the maximum error associated with the first-order perturbation theory
result is $O(10^{-3})$ and is often only $O(10^{-6})$.

Figure 3 shows our numerical results for another initial condition, consisting of
uniformly distributed random fluctuations of amplitude 0.75 about a uniform back-
ground of 4.5. Again, the analytic result (9) (denoted by a dashed line) is in very good
agreement with the numerical result (denoted by a solid line). Furthermore, the first-
order perturbation theory result (denoted by a dotted line in figure 3) is once again
almost indistinguishable from the numerical result. Figure 4 shows the absolute value
of the errors associated with our approximations for the simulation of figure 3.

At this stage, we should emphasize that the agreement of the numerical solution
with our analytic form improves progressively with time and the agreement is even
better at times later than those shown in figures 1-4. This is because our initial assump-
tion about e^{-t} is exact asymptotically in time. Furthermore, both solutions are rapidly
attracted to the same homogenous solution (namely equation (10)) and the amplitude
of fluctuations about this homogenous solution decay sharply in time. Given our initial
assumption about the smallness of e^{-t} at large t, it may appear that our analytic form
would be inappropriate for the relatively early times shown in figures 1-4. However,
our numerical results for a wide range of initial data (representative examples of which
were shown) indicate that our analytic form is reasonable even for earlier times than
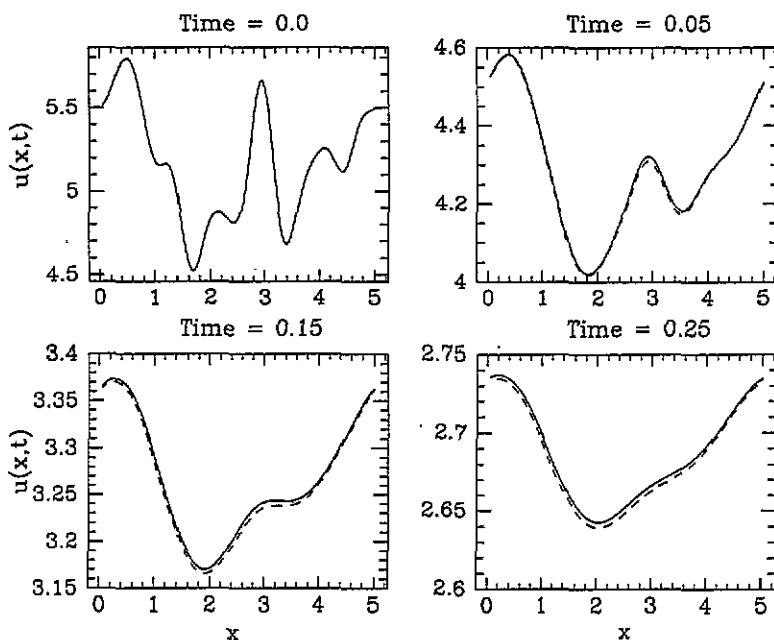the underlying assumption would suggest.

Figure 3. Similar to figure 1, except the initial condition (labelled by Time=0.0) now
consists of uniformly distributed random fluctuations of amplitude 0.75 about a background
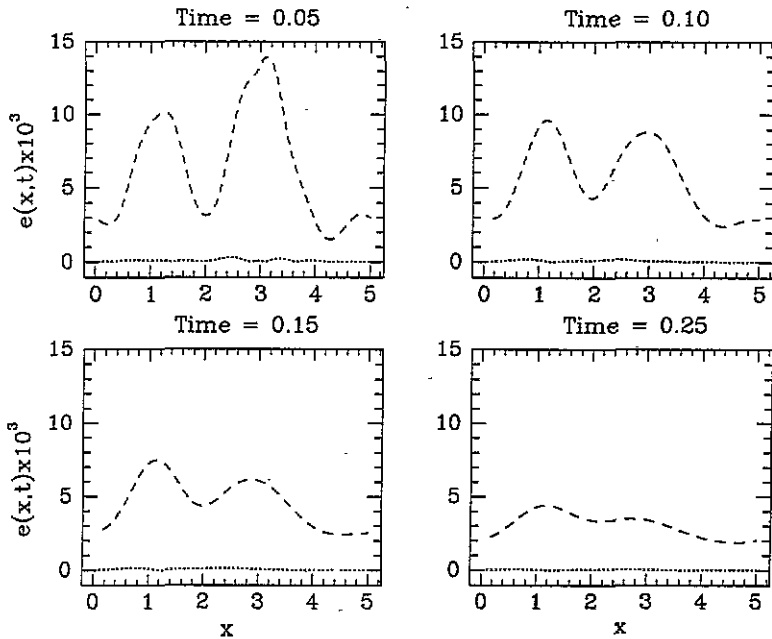value of 4.5. Symbols have the same meaning as in figure 1.

Figure 4. Similar to figure 2, except that the differences between the numerical simulation and the analytic results are for the simulation depicted in figure 3. Symbols have the same meaning as in figure 2.

As $t \to \infty$, both the numerical solution and our analytic form tend to $u^* = 1$. For initial conditions which lie sufficiently close to $u^* = 1$ so that we need consider only linear fluctuations about $u^* = 1$, our analytic form (9) becomes

$$u(r, t) \simeq 1 - e^{-t} e^{t\nabla^2}[1 - u(r, 0)]. \tag{16}$$

But this is identical to the solution obtained directly by considering only linear fluctuations around $u^* = 1$ for the Fisher equation, suggesting that our analytic form is exact as $t \to \infty$.

For completeness, we point out that we have also compared our numerical results for the Fisher equation with

(i) the analytic expression from singular perturbation theory as in (3) and a second-order perturbation theory around it; and

(ii) the analytic expression from a fourth-order perturbation theory around the uniformly translating homogenous solution in (10) of the Fisher equation.

In each of the above cases, the agreement with the numerical solution is so bad that we do not even show these comparisons on our figures.

As we remarked earlier, our methodology is applicable to other reaction–diffusion equations also. We have also applied it to the TDGL equation, namely

$$u_t(r, t) = u(r, t) - u(r, t)^3 + \nabla^2 u(r, t). \tag{17}$$

For the TDGL equation, the nonlinear transformation analogous to (4) is

$$u(r, t) = \frac{e^t f(r, t)}{[1 + [e^t f(r, t)]^2]^{1/2}} \tag{18}$$

which is also motivated by the corresponding singular perturbation result [8]. The nonlinear transformation is such that we confine ourselves to solutions for which $|u(r, t)| < 1$ for all finite times. A similar sequence of steps as before leads us to the analytical result for the TDGL equation (analogous to (9))

$$u(r, t) = \pm \frac{e^t}{\left[\left\{e^{t\nabla^2}\left(\frac{1 - u(r, 0)^2}{u(r, 0)^2}\right)\right\} + e^{2t}\right]^{1/2}} \tag{19}$$

In this case, the quantity which satisfies the diffusion equation is $1/f(r, t)^2$ and not $1/f(r, t)$, as before. Furthermore, the initial conditions are restricted to lie uniformly between 0 and $+1$ or 0 and -1 and the corresponding sign for the solution in (19) is $+1$ or -1. For this class of initial conditions, the numerical results are again in excellent agreement with the analytic result from (19). Similar analytic solutions can be derived for other reaction–diffusion equations also but we do not present these here as we have already amply illustrated our methodology.

To summarize: we have presented a novel method of asymptotically transforming the problem of solution of the initial value problem for the Fisher equation to the problem of solution of the diffusion equation for a class of initial conditions. Our method is valid for initial conditions which lie uniformly above 0 and gives excellent agreement with numerical simulations of the Fisher equation, even for relatively early times. If we improve our analytic approximation by a first-order perturbation expansion, the agreement with the numerical result is almost perfect. Our method easily generalizes to other reaction–diffusion equations and we have also presented an analytic result for the time-dependent Ginzburg–Landau equation, again for a restricted class of initial conditions.

Acknowledgments

References

[1] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform (SIAM Studies in Applied Mathematics)* (Philadelphia, PA: SIAM) and references therein
[2] Kuramoto Y 1984 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer) and references therein
[3] Hopf E 1950 *Commun. Pure Appl. Math.* 3 201
Cole J D 1951 *Quart. Appl. Math.* 9 225
[4] Burgers J 1974 *The Nonlinear Diffusion Equation* (Dordrecht: Reidel)
[5] Fisher R A 1937 *Ann. Eugenics* 7 355
Kolmogorov A, Petrovskii I and Piscounov N 1937 *Bull. Univ. Moscow Ser. Int. Sec.* A 1 1
[6] Ma S K 1981 *Modern Theory of Critical Phenomena* (Reading, MA: Benjamin-Cummins)
[7] Aronson D G and Weinberger H F 1975 *Partial Differential Equations and Related Topics* ed J A Goldstein (Berlin: Springer) pp 5–49
Aronson D G and Weinberger H F 1978 *Adv. Math.* 30 33
[8] Kawasaki K, Yalabik M C and Gunton J D 1978 *Phys. Rev.* A 17 455
[9] Puri S 1992 *Phys. Lett.* 164A 211
[10] Bray A J and Puri S 1991 *Phys. Rev. Lett.* 67 2670
[11] Puri S, Elder K R and Desai R C 1989 *Phys. Lett.* 142A 357
Puri S 1991 *Phys. Rev.* A 43 7031